

What is a point?

In the Zariski topology, a point is $\text{Spec } k$, for some field k . In the étale topology, separable extensions plays havoc with this. Now we know that $Y \rightarrow \text{Spec } \bar{k}$ is étale iff Y is a disjoint union of $\text{Spec } \bar{k}$. This is the notion we want, in analogy with covering spaces.

Def: "The Étale local ring" at x is: $\varinjlim \mathcal{O}_x(U)$, with $U \rightarrow X$ an étale map with x in the image, and the limit is over all pairs $(U, y \mapsto x)$.

Example: $X = \text{Spec } k$. Then $\varinjlim_{\substack{U/k \text{ fin.} \\ \text{sep.}}} U = \text{Spec } k$.
 $\begin{array}{c} \varinjlim \\ \uparrow \\ y \end{array}$ "marked point"

Taking the limit: $\varinjlim \mathcal{O}_x(U) = \varinjlim U/k = \bar{k}_{\text{sep}}$, i.e., the geometric point!

So in the étale topology, a point is then a geometric point. So we define our notion "more" correctly:

Def: The étale local ring at a point \bar{x} is $\mathcal{O}_{x, \bar{x}} = \varinjlim \mathcal{O}_x(U)$, where the limit is over pairs (U, i) , with:

$\begin{array}{ccc} \bar{x} & \xrightarrow{i} & U \\ & \searrow & \downarrow \text{étale} \\ & & X \end{array}$ | This is called the strict henselization of the local ring $\mathcal{O}_{x, x}$.

Def: A local ring (A, \mathfrak{m}) is henselian if the ring satisfies Hensel's lemma. That is, for all $f(t) \in A[t]$, with $\bar{f}(t) = g_0(t) \cdot h_0(t)$ in $A/\mathfrak{m}[t]$ with g_0, h_0 monic & coprime, there is $g, h \in A[t]$ monic and $\bar{g} = g_0, \bar{h} = h_0$.

As a remark, $(g, h) = A[t]$, as $A[t]/(g(t))$ is a f.g. A -module $\Rightarrow A[t]/(g, h)$ is a f.g. A -module, M . Thus since \bar{g}, \bar{h} are coprime:

$$M/\mathfrak{m}M = A/\mathfrak{m}[t]/(\bar{g}, \bar{h}) = 0.$$

So $M=0$ by Nakayama. Hence g and h are coprime, and we can use this to show uniqueness. See any text for the proof of this (or do it yourself).

Exercise 13:(5) Prove Theorem 4.2 in EC.

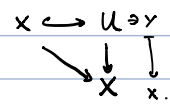
In particular, the above states if (A, \mathfrak{m}) is henselian, $X = \text{Spec } A$, x the closed point, then if $f: Y \rightarrow X$ is étale & $\exists y \in Y$ w/ $y \mapsto x$ with $k(y) \cong k(x)$, then f has a section.

Def: Let (A, \mathfrak{m}) be a local ring, and $A \xrightarrow{f} A^h$ is a (local) morphism, then A^h is the henselization of A if for all $A \rightarrow B$, with B henselian, there is a unique morphism making

$$\begin{array}{ccc} A & \xrightarrow{f} & A^h \\ & \searrow & \downarrow \exists! \\ & & B \end{array}$$

commute.

Exercise (5): Let $x \in X$, $A = \mathcal{O}_{X,x}$. Take $\tilde{A} = \varinjlim \mathcal{O}(U)$, with



Then $\mathcal{O}_{X,x} \rightarrow \tilde{A}$ is the henselization.

Prop: Let (A, \mathfrak{m}) be a complete local ring ($A \xrightarrow{\sim} \varprojlim A/\mathfrak{m}^n$). Then A is henselian.

Proof: Prop. 4.5. \square

This allows us to construct the henselization. Take $A = \mathcal{O}_{X,x}$, then we have an injective map $A \hookrightarrow \hat{A}$ (the completion, which is henselian by above). Now take: $A^h = \bigcap B$, over $A \subset B$, B henselian.

Example: Take $k[t_1, \dots, t_n]_{(0)} \subset k[[t_1, \dots, t_n]]$. A theorem of Artin says the former ring is all algebraic power series.

Thm: Let (A, \mathfrak{m}) , $k = A/\mathfrak{m}$, be henselian. Then there is an equivalence of categories $\{\text{FET}/A\} \leftrightarrow \{\text{FET}/k\}$.

This reasonable, by sending $B/A \rightsquigarrow B \otimes_A k/k$. The details are technical.

Étale Fundamental Group

By analogy with covering spaces, we can give an algebraic construction of the fundamental group.

Def: Let $Y \xrightarrow{\text{connected}} X$ be a finite étale morphism. It is called Galois with Galois group G (with $|G| < \infty$) if $G \rightarrow \text{Aut}(Y/X)$. That is, we have an action map $G \times Y \xrightarrow{\sim} Y$ s.t. $G \times_x Y \xrightarrow{\sim} Y \times_x Y$.

Def: Given any X , we define the étale fundamental group $\pi_1(X) = \varprojlim G$ with G a Galois group $Y \xrightarrow{G} X$.

See EC for examples.