What is a point?

In the Zariski topology, a point is Spec k, for some field k. In the étale topology, separable extensions plays havoc with this. Now we know that $Y \to \operatorname{Spec} \bar{k}$ is étale iff Y is a disjoint union of $\operatorname{Spec} \bar{k}$. This is the notion we want, in analogy with covering spaces.

Def: "The Étale local ring" at x is: $\varinjlim \mathcal{O}_X(U)$, with $U \to X$ an étale map with x in the image, and the limit is over all pairs $(U, y \mapsto x)$.

Example: $X = \operatorname{Spec} k$. Then $\coprod_{L/k \text{ fin. sep.}} L = U \longrightarrow \operatorname{Spec} k$.

$$\begin{array}{c} U \\ \Psi \\ y \end{array}$$

$\nwarrow$ "marked point"

Taking the limit: $\varinjlim \mathcal{O}_X(U) = \lim_{\text{fin. sep}} L/k = \bar{k}_{\text{sep.}}$, i.e., the geometric point!

So in the étale topology, a point is then a geometric point. So we define our notion "more" correctly:

Def: The étale local ring at a point $\bar{x}$ is $\mathcal{O}_{X,\bar{x}} = \varinjlim \mathcal{O}_X(U)$, where the limit is over pairs (U, i), with:

$$\begin{tikzcd} \bar{x} \arrow[r, hook, "i"] \arrow[dr] & U \arrow[d, "\text{étale}"] \\ & X \end{tikzcd}$$

This is called the strict henselization of the local ring $\mathcal{O}_{X,x}$.

Def: A local ring (A, m) is henselian if the ring satisfies Hensel's lemma. That is, for all $f(t) \in A[t]$, with $\bar{f}(t) = g_0(t) \cdot h_0(t)$ in $A/m[t]$ with g_0, h_0 monic & coprime, there is $g, h \in A[t]$ monic and $\bar{g} = g_0$, $\bar{h} = h_0$.

As a remark, $(g, h) = A[t]$, as $A[t]/(g(t))$ is a f.g. A-module $\Rightarrow$ $A[t]/(g,h)$ is a f.g. A-module, M. Thus since $\bar{g}, \bar{h}$ are coprime:

$$M/mM = {A/m[t]}/{(\bar{g}, \bar{h})} = 0.$$

So $M = 0$ by Nakayama. Hence g and h are coprime, and we can use this to show uniqueness. See any text for the proof of this (or do it yourself).

Exercise 13: (5) Prove Theorem 4.2 in EC.

In particular, the above states if (A, m) is henselian, $X = \operatorname{Spec} A$, x the closed point, then if $f: Y \to X$ is étale & $\exists\, y \in Y$ w/ $y \mapsto x$ with $k(y) \cong k(x)$, then f has a section.

Def: Let (A, m) be a local ring, and $A \xrightarrow{f} A^h$ is a (local) morphism, then A^h is the henselization of A if for all $A \to B$, with B henselian, there is a unique morphism making

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr] & A^h \arrow[d, dashed, "\exists!"] \\ & B \end{tikzcd}$$

commute.

Exercise (5): Let $x \in X$, $A = \mathcal{O}_{X,x}$. Take $\tilde{A} = \varinjlim \mathcal{O}(U)$, with
$$\begin{tikzcd} x \arrow[r, hook] & U \ni y \arrow[d] \\ & X \arrow[from=1-1, to=2-2] & x. \end{tikzcd}$$
Then $\mathcal{O}_{X,x} \longrightarrow \tilde{A}$ is the henselization.

Prop: Let (A, m) be a complete local ring $(A \xrightarrow{\sim} \varprojlim A/m^n)$. Then A is henselian.
Proof: Prop. 4.5. ▨

This allows us to construct the henselization. Take $A = \mathcal{O}_{X,x}$, then we have an injective map $A \hookrightarrow \hat{A}$ (the completion, which is henselian by above). Now take: $A^h = \bigcap B$, over $A \subset B$, B henselian.

Example: Take $k[t_1, \dots, t_n]_{(0)} \subset k[[t_1, \dots, t_n]]$. A theorem of Artin says the former ring is all algebraic power series.

Thm: Let (A, m), $k = A/m$, be henselian. Then there is an equivalence of categories $\{FEt/A\} \longleftrightarrow \{FEt/k\}$.

This reasonable, by sending $B/A \rightsquigarrow B \otimes_A k/k$. The details are technical.

Étale Fundamental Group

By analogy with covering spaces, we can give an algebraic construction of the fundamental group.

Def: Let $Y \longrightarrow X$ (connected) be a finite étale morphism. It is called Galois with Galois group G (with $|G| < \infty$) if $G \longrightarrow \mathrm{Aut}(Y/X)$. That is, we have an action map $G \times Y \rightrightarrows Y$ s.t. $G \times_X Y \xrightarrow{\sim} Y \times_X Y$.

Def: Given any X, we define the étale fundamental group $\pi_1(X) = \varprojlim G$ with G a galois group $Y \xrightarrow{G} X$.

See EC for examples.